Fermionic functional renormalization group approach to strongly coupled superfluidity

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1. INTRODUCTION

- In the past decade strongly coupled fermionic superfluidity has been studied experimentally via cold atomic experiments, due to its importance in astro- and condensed matter physics.
- The understanding of strongly coupled systems requires the development of non-pertubative theoretical methods.
- We wish to attack this problem within the Functional Renormalization Group (FRG) approach [1, 2].

Model: two-component fermionic systems with a contact interaction:

$$S = \int_0^\beta \mathrm{d}\tau \int \mathrm{d}^3\boldsymbol{x} \left[\overline{\psi} \left(\partial_\tau - \frac{\nabla^2}{2m} - \mu \right) \psi + g \overline{\psi}_\uparrow \overline{\psi}_\downarrow \psi_\downarrow \psi_\uparrow \right]$$

(β^{-1}: temperature, m: mass, μ: chemical potential, g: bare coupling)

Basic strategy: FRG introduces an IR regulated effective action Γ_k including fluctuations with energy $\gtrsim k^2/2m$. This action interpolates between classical action S ($k \approx \infty$) and quantum action Γ ($k = 0$):

$$\partial_k \Gamma_k[\overline{\psi}, \psi] = \frac{1}{2} \mathrm{STr} \left[\left(\Gamma_k^{(2)}[\overline{\psi}, \psi] + R_k \right)^{-1} \partial_k R_k \right]$$

($\Gamma_k^{(2)}[\overline{\psi}, \psi]$: field-dependent propagator, R_k: infrared regulator).

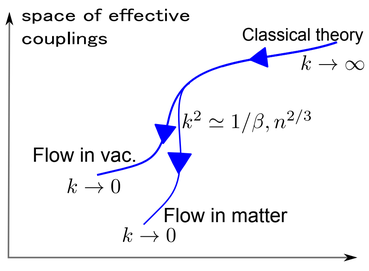
Goals:

- Clarify the relation between the strongly-coupled theory and the conventional wisdom, i.e. the BCS theory [3] with Gorkov and Melik-Barkhudarov (GMB) correction [4] for weak couplings.
- Develop fermionic FRG in a systematic, improvable way without the help of auxiliary bosonic fields.
- Calculate the critical temperature as a function of the scattering length ($a_S < 0$), especially in the unitarity regime ($a_S \simeq \pm\infty$), which is not yet fully understood [5].

2. FERMIONIC FUNCTIONAL RG

Why fermionic FRG?

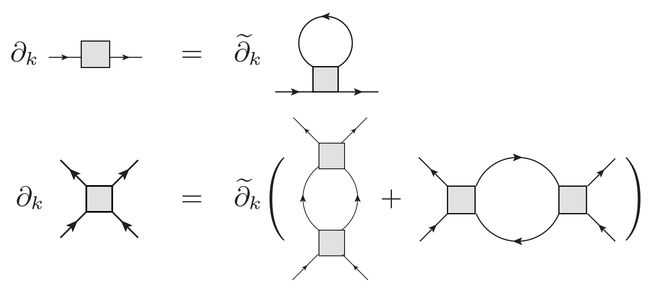
- Less-biased study of non-perturbative physics.
- More natural to apply improvable approximations.
- Scale separation enables to observe the physical origin of gradual corrections.



Our approximation contains the self energy Σ_k ($\Gamma_k^{(2)} = G^{-1} - \Sigma_k$) and the four-point vertex $\Gamma_k^{(4)}$:

$$\Gamma_k[\bar{\psi}, \psi] = \sum\!\!\!\!\!\!\int_p \overline{\psi}_p [G^{-1} - \Sigma_k](p) \psi_p + \sum\!\!\!\!\!\!\int_{p,q,q'} \Gamma_k^{(4)}(p; q, q') \overline{\psi}_{\uparrow, p/2+q} \overline{\psi}_{\downarrow, p/2-q} \psi_{\downarrow, p/2-q'} \psi_{\uparrow, p/2+q'}.$$

Diagrammatic representation of flow equations of Σ_k and $\Gamma_k^{(4)}$:



The flow of $\Gamma_k^{(4)}$ consists of particle-particle (PP) and particle-hole (PH) fluctuations. (Boxes represent 1PI vertices, $\widetilde{\partial}_k = \partial_k R_k \partial / \partial R_k$.)

3. RELATION WITH BCS THEORY

- In this part, we neglect the self-energy correction: $\Sigma_k = 0$
- We keep only particle-particle fluctuations in the flow of $\Gamma_k^{(4)}$
- These assumptions exactly reproduce BCS theory

Flow equation of $\Gamma_k^{(4)}$ simplifes as:

$$\partial_k \Gamma_k^{(4)}(p) = \widetilde{\partial}_k \sum\!\!\!\!\!\!\int_l \frac{-\Gamma_k^{(4)}(p) \Gamma_k^{(4)}(p)}{[G^{-1} + R_k]\left(\frac{p}{2} + l\right)[G^{-1} + R_k]\left(\frac{p}{2} - l\right)}.$$

The solution also corresponds to PP-RPA (Random Phase Approximation):

$$\frac{1}{\Gamma_k^{(4)}(p)} = \frac{1}{g} + \sum\!\!\!\!\!\!\int_l \frac{1}{[G^{-1} + R_k]\left(\frac{p}{2} + l\right)[G^{-1} + R_k]\left(\frac{p}{2} - l\right)}.$$

- Renormalization condition: $\Gamma_{k=0}^{(4)}|_{T=0,\mu=0} = m/4\pi a_S$
- Thouless criterion: $[\Gamma_{k=0}^{(4)}(p=0)]^{-1}\big|_{T=T_c} = 0$
- These lead to the well-known BCS transition temperature: (ε_F: Fermi energy, k_F: Fermi momentum)

$$T_c^{BCS}/\varepsilon_F = \frac{8}{\pi} e^{\gamma_E - 2} \exp(-\pi/2k_F|a_S|).$$

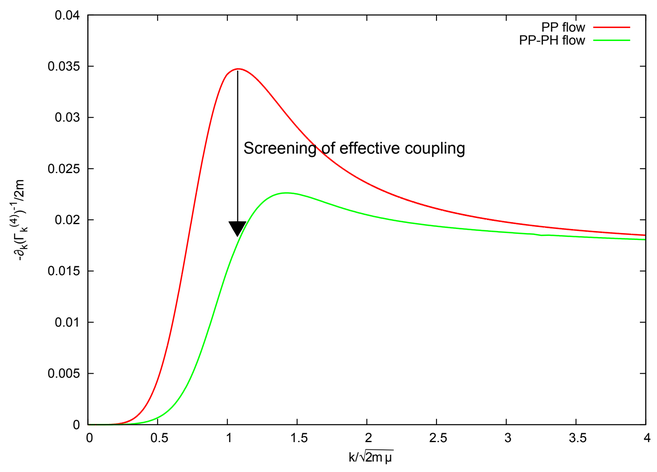
4. GORKOV CORRECTION WITHIN FRG

- Inclusion of particle-hole fluctuations leads to a screening of the effective coupling in matter.
- Assumptions on the particle-hole loop:
  → relative momentum dependence of $\Gamma_k^{(4)}$ is neglected,
  → concerning the propagators, relative momenta q, q' are projected onto the Fermi sphere.

The particle-hole contribution of the flow equation of $1/\Gamma_k^{(4)}$ at $p = 0$:

$$\widetilde{\partial}_k \int_{|\boldsymbol{q}|,|\boldsymbol{q}'|=k_F} \sum\!\!\!\!\!\!\int_l \frac{1}{[G^{-1} + R_k](l)[G^{-1} + R_k][q - q' + l]}$$

This term changes the flow of PP-RPA (red curve) to the flow describing the GMB correction (green curve).



Screening effect leads to reduction of T_c (GMB correction):

$$T_c^{GMB} = T_c^{BCS}/(4e)^{1/3}.$$

5. EFFECT OF SELF-ENERGY

- A natural step forward is to include the effect of the self-energy
- We fully neglect its momentum dependence: $\Sigma_k(p) = \Sigma_k$

Flow equation of Σ_k is obtained as:

$$\partial_k \Sigma_k = \widetilde{\partial}_k \sum\!\!\!\!\!\!\int_l \frac{\Gamma_k^{(4)}(l)}{[G^{-1} - \Sigma_k + R_k](l)}.$$

Neglecting the momentum dependence of $\Gamma_k^{(4)}$ and then solving the coupled equations of Σ_k and $\Gamma_k^{(4)}(0)$ leads to:

- a rather small self-energy and correction of the Fermi-surface,
- a rather close value of T_c compared to T_c^{GMB},
- Wrong asymptotic k behavior of Σ_k as $k \to \infty$
  $\Longrightarrow \Sigma_k \sim k^2 \exp(-k^2/2mT)$.

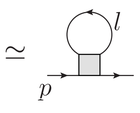
It is believed that approximations fail to reproduce correct large-k behavior might not be reasonable, therefore further correction is needed.

6. CORRECTED Σ_k FLOW

- We include the momentum dependence of $\Gamma_k^{(4)}$
- This leads to a correct self-energy flow at large-k

What is the correct large-k behavior?
⟶ one-loop approximation should be appropriate if k is large

$$\Sigma_k \simeq \quad =\sum\!\!\!\!\!\!\int_l \frac{\Gamma_k^{(4)}(l)}{[G^{-1} - \Sigma_k + R_k](l)} \simeq \frac{k_F^3}{4mk}$$



⇒ Unlike the exponential suppression, this already gives a non-negligible contribution when $k \sim \mathcal{O}(k_F)$.

Non-local property of $\Gamma_k^{(4)}$ plays an important role. We approximate it in the flow equation of Σ_k as:

$$\Gamma_k^{(4)}(p) = -\frac{Z_k^{-1}}{ip^0 + \omega_k^B(\boldsymbol{p})},$$

which is the Taylor series of $1/\Gamma_k^{(4)}$ around zero momenta. Z_k^{-1} and $\omega^B(\boldsymbol{p})$ depends on various derivatives of $\Gamma_k^{(4)}$, these are calculated within the PP Random Phase Approximation of $\Gamma_k^{(4)}$.
$\Longrightarrow$ large-k asymptotics turn out to be the correct one!

Since the lowest frequency of Σ_k is actually $\pm\pi T$, we arrive at

$$\partial_k \Sigma_k = \widetilde{\partial}_k \sum\!\!\!\!\!\!\int_l \frac{1}{[G^{-1} - \Sigma_k + R_k](l)} \frac{-Z_k^{-1}}{i(l^0 \pm \pi T) + \omega_k^B(\boldsymbol{l})}.$$

- We take into account effective bosonic excitations via the momentum dependence of $\Gamma_k^{(4)}$ (i.e. bosonic propagator appears).
- This has a huge effect on the number equation as well: $n = \int (G^{-1} - \Sigma_{k=0})^{-1}$, through which ε_F is defined.

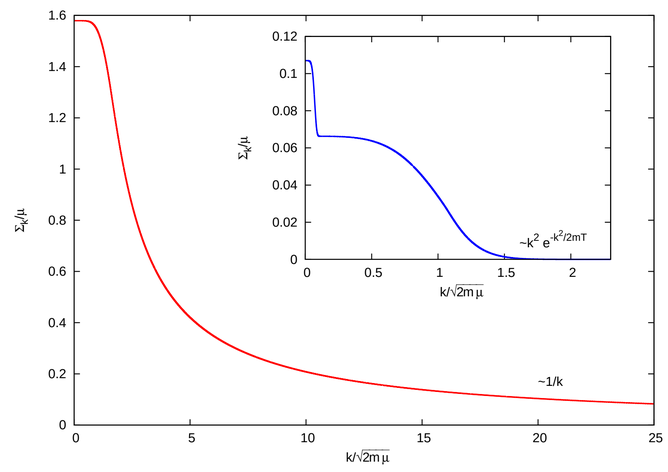
7. NUMERICS

- For numerical purpose, we choose to use unit system $2m = 1$.
- The flow of Σ_k and Γ_k^{-1} are obtained on a grid with a step size of $\Delta k = 10^{-4}$ in an interval $I = [0, 50]$.
- Instead of running the flows from a UV scale to zero, we start from $k = 0$ and obtain the solutions up to $\Lambda_{UV} = 50$.
- Boundary conditions: $\Gamma_{k=0}^{-1} = 0$, $\Sigma_{k=\infty} = 0$.
- The full solution of the functions are obtained by fitting appropriate asymptotics in an interval of $\tilde{I} = [49, 50]$.

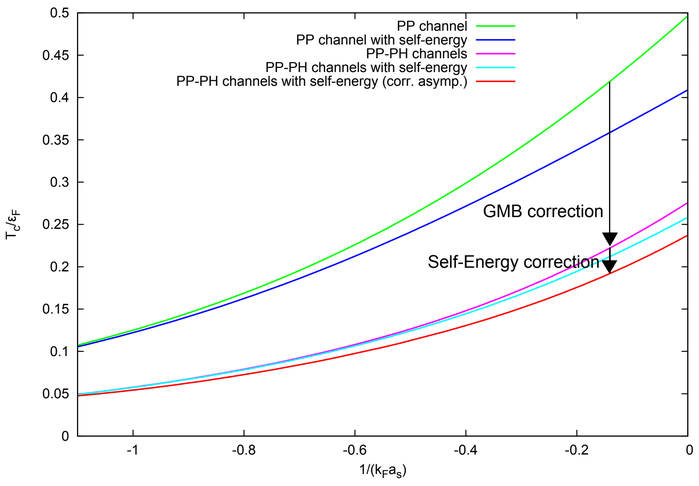
8. RESULTS

	T_c/ε_F	μ/ε_F	T_c/μ
PP fluctuations, no self-energy	0.496	0.747	0.665
PP fluctuations with self-energy	0.409	0.601	0.680
PP+PH fluctuations, no self-energy	0.276	0.929	0.297
PP+PH fluctuations with self-energy	0.255	0.849	0.300
PP+PH fluct. with corrected self-energy	**0.237**	**0.367**	**0.646**

- Flow of self-energy at $a_S = \infty$ in the two corresponding cases:



- Transition temperature as a function of the inverse scattering length for various approximations:



We observe a chain of gradual changes in the critical temperature as we improve approximation schemes:

- In the weak-coupling region $(k_F a_S)^{-1} \lesssim -1$, GMB correction provides the dominant correction from BCS theory.
- As we approach to unitarity $(k_F a_S)^{-1} = 0$, self-energy correction also provides a significant, almost 20% correction.

9. CONCLUSIONS & REFERENCES

- We developed a systematically improvable approximation scheme within the fermionic functional renormalization group.
- The set of approximations include
  1. PP-RPA reproducing the BCS theory,
  2. the GMB correction, by taking into account PH loop in the effective coupling flow,
  3. self-energy correction within a local approximation,
  4. its correction due to effective composite bosons.
- Analyzing the flow corresponding to each scheme we argued that due to PH fluctuations, a screening of the effective coupling occurs at an intermediate energy scale $k \sim k_F$.
- To get a correct self-energy correction Σ_k, one has to take into account bosonic fluctuations through the effective coupling.
- In the latter case, Σ_k is not neglegible, it is still large around the region $k/k_F \gtrsim$ 2-3 of the flow.
- At the unitary point $a_S = \infty$ we got the following values:
  ⟶ $\boldsymbol{T_c/\varepsilon_F = 0.237}$, $\boldsymbol{\mu/\varepsilon_F = 0.367}$.

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